

INVARIANT POLYNOMIALS OF THE AUTOMORPHISM GROUP OF A COMPACT COMPLEX MANIFOLD

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1. Introduction

Let M be a compact complex manifold of dimension n , $H(M)$ the complex Lie group of all automorphisms of M , and $\mathfrak{h}(M)$ the complex Lie algebra of all holomorphic vector fields of M . When $c_1(M)$ is positive, the first author defined in [13] a character $f: \mathfrak{h}(M) \rightarrow \mathbf{C}$ which is intrinsically defined, vanishes if M admits a Kähler-Einstein metric, and has its origin in Kazdan-Warner's integrability condition for Nirenberg's problem [16].

In this note we give a better understanding of f along the lines of the classical works by Bott and the recent works in symplectic geometry by Duistermaat-Heckman [12], Berline-Vergne [3, 4], and Atiyah-Bott [1]. We begin by rephrasing Theorem 2.18 of Berline-Vergne [3] in the following way; there exists a linear map $F: I^{n+k}(\mathrm{GL}(n, \mathbf{C})) \rightarrow I^k(\mathfrak{h}(M))$ where, for a complex Lie group G , $I^p(G)$ denotes the set of all holomorphic G -invariant symmetric polynomials of degree p . The character f coincides with $F(c_1^{n+1})$ up to a constant. By a proof identical to Bott [5, 6] we have a localization formula of the elements of the image of F . The main result of this note is to show explicitly that the linear map F corresponds to the Gysin map in the context of equivariant cohomology (Theorem 4.1 and Corollary 4.2).

We also give another interpretation of f in terms of secondary characteristic classes of Chern-Simons [11] and Cheeger-Simons [10]. More precisely we find that f appears as the so-called Godbillon-Vey invariant of certain complex foliations which are defined naturally.

The linear map F , which depends only on the complex structure of M , may be regarded as a generalization of f . There is another type of generalization of f ([14], [9], [2]) which depends on a fixed Kähler class. We think that this latter one also deserves further study.

2. Definition of F

Let P be a complex analytic fiber bundle over M with the right action of a complex Lie group G . Suppose $H(M)$ acts on P from the left complex analytically and commuting with the action of G . Let θ be any type $(1,0)$ connection and Θ the curvature form of θ . Since $H(M)$ acts on P , $X \in \mathfrak{h}(M)$ defines a vector field on P , which we shall denote by the same letter X . Then since $\phi(-\theta(X) + \frac{i}{2\pi}\Theta)$, $\phi \in I^{n+k}(G)$, is horizontal and G -invariant it projects to a form on M . We define

$$f_\phi(X) = \int_M \phi\left(-\theta(X) + \frac{i}{2\pi}\Theta\right).$$

The following is a complex version of Theorem 2.18 of [3] and is proved similarly.

Proposition 2.1. *The definition of f_ϕ does not depend on the choice of the type $(1,0)$ connection θ . Furthermore f_ϕ is invariant under the coadjoint action of $H(M)$. So we obtain a linear map $F: I^{n+k}(G) \rightarrow I^k(H(M))$.*

Let θ be a type $(1,0)$ connection of the holomorphic tangent bundle of M which is associated by the frame bundle of M . Let D be the covariant differentiation and put $L(X) = L_X - D_X$ for $X \in \mathfrak{h}(M)$ where L_X is the Lie differentiation by X . For $\phi \in I^{n+k}(\mathrm{GL}(n, \mathbf{C}))$ we define

$$f_\phi(X) = \int_M \phi\left(-L(X) + \frac{i}{2\pi}\Theta\right).$$

From Proposition 2.1 together with Lemma 1.10 in [4] we obtain the same conclusion for the new f_ϕ with $G = \mathrm{GL}(n, \mathbf{C})$. This conclusion also follows from Bott's localization theorem. We say that X is nondegenerate if zeros of X are isolated and if at each zero p the linear map $L(X)_p: T_p M \rightarrow T_p M$ is nondegenerate.

Proposition 2.2 (Bott [5]). *If X is nondegenerate then*

$$f_\phi(X) = \sum_p \phi(L(X)_p) / \det L(X)_p.$$

Now we assume that $c_1(M)$ is positive. We put $c_1^+(M)$ to be the set of all positive $(1,1)$ forms representing $c_1(M)$. Choose any $\omega \in c_1^+(M)$ which is regarded as a Kähler form. Denote by γ_ω the Ricci form which also represents $c_1(M)$. Since $\gamma_\omega - \omega$ is a real exact $(1,1)$ form there exists a real smooth function F_ω , uniquely determined up to an additive constant, such that $\gamma_\omega - \omega = \frac{1}{2}(i/\pi)\partial\bar{\partial}F_\omega$. By definition ω is Kähler-Einstein iff F_ω is constant.

We define a linear function $f: h(M) \rightarrow \mathbb{C}$ by

$$f(X) = \int_M XF_\omega \omega^n.$$

In [13] we proved that the definition of f does not depend on the choice of $\omega \in c_1^+(M)$.

Proposition 2.3. $f_{c_1^{n+1}} = (n + 1)f$.

Proof. By the Calabi-Yau theorem [18] there exists a unique Kähler form $\eta \in c_1^+(M)$ such that $\gamma_\eta = \omega$. Therefore we may assume $F_\omega = -\log(\omega^n/\eta^n)$. It then follows from the divergence theorem with respect to the Kähler form η that

$$\begin{aligned} f(X) &= - \int_M X \log(\omega^n/\eta^n) \omega^n = - \int_M X(\gamma_\eta^n/\eta^n) \eta^n \\ &= - \int_M \delta'' X \gamma_\eta^n \\ &= \int_M \text{trace}(DX) \left(\text{trace} \left(\frac{i}{2\pi} \Theta \right) \right)^n \end{aligned}$$

where D and Θ is the covariant differentiation and the curvature form with respect to η and $\text{trace}(DX)$ makes sense because DX is a section of $T^*M \otimes TM \simeq \text{End}(TM)$. Since η is Kähler we have $L(X) = -DX$. This proves the proposition.

3. Secondary characteristic classes of complex foliations

First we review some known facts about complex foliations. Let W be a smooth manifold. A complex foliation \mathcal{F} of codimension q is an open covering $\{U_\alpha\}$ of W such that

- (1) there exist submersions $\gamma_\alpha: U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^q$,
- (2) on $U_\alpha \cap U_\beta \neq \emptyset$, $\gamma_{\alpha\beta} = \gamma_\alpha \circ \gamma_\beta^{-1}: \gamma_\beta(U_\alpha \cap U_\beta) \rightarrow \gamma_\alpha(U_\alpha \cap U_\beta)$ is a complex analytic diffeomorphism.

We may choose local coordinates $(t_\alpha^1, \dots, t_\alpha^p, x_\alpha^1, y_\alpha^1, \dots, x_\alpha^q, y_\alpha^q)$ on U_α so that $\gamma_\alpha(t_\alpha, x_\alpha, y_\alpha) = (z_\alpha^1, \dots, z_\alpha^q)$ where $z_\alpha^i = x_\alpha^i + iy_\alpha^i$. It is easy to observe that the covectors $dz_\alpha^i, i = 1, \dots, q$, span a well defined subbundle of $T^*W \otimes \mathbb{C}$ which we will denote by $T^*W^{1,0}$. We denote by $TW^{0,1}$ the subbundle of $TW \otimes \mathbb{C}$ spanned by vectors annihilated by the covectors in $T^*W^{1,0}$. Clearly $TW^{0,1}$ is spanned by vectors $\partial/\partial t_\alpha^i$ and $\partial/\partial \bar{z}_\alpha^i$. The quotient bundle $\nu(\mathcal{F}) = TW \otimes \mathbb{C}/TW^{0,1}$ is called the normal bundle of \mathcal{F} . A connection ∇ of $\nu(\mathcal{F})$ is called

a Bott connection if for $X \in TW^{0,1}$ and $Y \in C^\infty(\nu(\mathcal{F}))$ we have

$$(3.1) \quad \nabla_X Y = \pi[\tilde{X}, \tilde{Y}]$$

where $\tilde{X} \in C^\infty(TW^{0,1})$ is an arbitrary extension of X , $\tilde{Y} \in C^\infty(TW \otimes \mathbf{C})$ is an arbitrary lift of Y , and $\pi: TW \otimes \mathbf{C} \rightarrow \nu(\mathcal{F})$ is the projection. It is easy to check that this definition is well defined. Roughly speaking a Bott connection is a type (1, 0) connection. So by the type reasons:

Theorem (Bott [7]). *Let ∇ be a Bott connection and Θ the curvature form of ∇ . Then $\phi(\Theta) = 0$ for $\phi \in I^j(\text{GL}(n, \mathbf{C})), j > q$.*

Now we put $I'_0(\text{GL}(n, \mathbf{C})) = I^j(\text{GL}(n, \mathbf{C})) \cap \mathbf{Z}[c_1, \dots, c_n]$. By the argument of Cheeger-Simons [10] we can define a class $S_\phi(\mathcal{F}, \nabla) \in H^{2j-1}(W; \mathbf{C}/\mathbf{Z})$ for $\phi \in I'_0(\text{GL}(n, \mathbf{C})), j > q$. When $j = q + 1$, it is known that $S_\phi(\mathcal{F}, \nabla)$ is independent of the choice of the Bott connection; so we shall write it $S_\phi(\mathcal{F})$. And $S_{c_1^{n+1}}(\mathcal{F})$ is known as the Godbillon-Vey class.

Let M be a compact complex manifold and $W = M \times S^1$ where $S^1 = \mathbf{R}/\mathbf{Z}$. Consider a vector field $Y = \partial/\partial t + 2 \text{Re}(X)$ on W where $\text{Re}(X)$ is the real part of $X \in h(M)$ and t is the coordinate of S^1 . Then the flow generated by Y defines a complex foliation \mathcal{F} of codimension n .

Theorem 3.1. *For any $\phi \in I_0^{n+1}(\text{GL}(n, \mathbf{C}))$ we have*

$$S_\phi(\mathcal{F})[W] = \frac{i}{2\pi} f_\phi(X) \text{ mod } \mathbf{Z}.$$

Proof. We denote by \mathcal{F}_λ the foliation obtained by replacing Y by $Y_\lambda = \partial/\partial t + 2 \text{Re}(\lambda X)$ for any $\lambda \in \mathbf{R}$. Then we have $\nu(\mathcal{F}_\lambda) \simeq \pi^*TM$ where $\pi: M \times S^1 \rightarrow M$ and TM is the holomorphic tangent bundle of M . Let h be any Hermitian metric of TM and D its connection. We define a Bott connection ∇^λ by

$$(3.2) \quad \begin{aligned} \nabla_{\partial/\partial z^j}^\lambda \frac{\partial}{\partial z^j} &= (\pi^*D)_{\partial/\partial z^j} \frac{\partial}{\partial z^j}, & \nabla_{\partial/\partial \bar{z}^j}^\lambda \frac{\partial}{\partial z^j} &= 0, \\ \nabla_{Y_\lambda}^\lambda \frac{\partial}{\partial z^j} &= \pi_* \left[Y_\lambda, \frac{\partial}{\partial z^j} \right] = \lambda \left[X, \frac{\partial}{\partial z^j} \right]. \end{aligned}$$

Then from (3.2) we have

$$(3.3) \quad \nabla_{\partial/\partial t}^\lambda \frac{\partial}{\partial z^j} = \nabla_{Y_\lambda - \lambda X}^\lambda \frac{\partial}{\partial z^j} = \lambda L(X) \frac{\partial}{\partial z^j}.$$

Denoting by θ^λ and θ the connection forms of ∇^λ and D

$$(3.4) \quad \frac{d}{d\lambda} \theta^\lambda = \frac{d}{d\lambda} (\pi^* \theta + \lambda L(X) dt) = L(X) dt.$$

Moreover the curvature form Θ^λ of θ^λ is computed as

$$(3.5) \quad \Theta^\lambda = d\theta^\lambda + \theta^\lambda \wedge \theta^\lambda = \pi^* \Theta^\lambda \text{ mod } dt.$$

It follows from (3.4), (3.5), and Proposition 2.9 in [10] that

$$\begin{aligned}
 \frac{d}{d\lambda} S_\phi(\mathcal{F}_\lambda)[W] &= (n+1) \int_W \phi\left(\frac{i}{2\pi} \frac{d}{d\lambda} \theta^\lambda, \frac{i}{2\pi} \Theta^\lambda, \dots, \frac{i}{2\pi} \Theta^\lambda\right) \bmod \mathbf{Z} \\
 (3.6) \qquad &= \frac{(n+1)i}{2\pi} \int_M \phi\left(L(X), \frac{i}{2\pi} \Theta, \dots, \frac{i}{2\pi} \Theta\right) \bmod \mathbf{Z} \\
 &= \frac{1}{2\pi i} f_\phi(X) \bmod \mathbf{Z}.
 \end{aligned}$$

Since the right-hand side does not depend on λ we obtain Theorem 3.2 by integrating the both sides of (3.6) over $[0, 1]$ with respect to λ .

4. Relation to equivariant cohomology

For brevity we shall write H for $H(M)$. Let $EH \rightarrow BH$ be the universal H -bundle. We put $MH = EH \times_H M$. Let P be as in §2. Then $PH = EH \times_H P$ is a principal G -bundle over MH .

Theorem 4.1. *The following diagram commutes:*

$$\begin{array}{ccc}
 I^{n+k}(G) & \xrightarrow{\Phi = (i/2\pi)^k F} & I^k(H) \\
 \downarrow W & & \downarrow W \\
 H^{2n+2k}(MH) & \xrightarrow{\pi_*} & H^{2k}(BH)
 \end{array}$$

where two W 's are Weil maps corresponding to $PH \rightarrow MH$ and $EH \rightarrow BH$, and π_* is the Gysin map of $\pi: MH \rightarrow BH$.

Proof. We may prove it for a principal H -bundle E over a finite-dimensional base space B instead of $EH \rightarrow BH$. Let κ be a connection form of $E \rightarrow B$ and V the horizontal distribution. Let $X_\#$ be a right invariant horizontal (local) vector field of E .

Lemma 4.2. $X_\#$ defines a well-defined vector field X on PH . In particular V defines a distribution V' in $T(PH)$ whose dimension is equal to $\dim B$.

Proof. Let ξ_t be the flow generated by $X_\#$. Then by the right invariance of $X_\#$ we have $\xi_t(eh) = \xi_t(e)h$ for any $e \in P$ and $h \in H$. We put $X = (d/dt)(\xi_t(e), p) \in T(PH)$, $p \in P$. This is well defined because

$$(\xi_t(eh), h^{-1}p) = (\xi_t(e)h, h^{-1}p) = (\xi_t(e), p). \qquad \text{q.e.d.}$$

Let $\pi_2: PH \rightarrow B$ be the projection and $T(\pi_2)$ the vector bundle consisting of all vectors tangent to the fibers of π_2 . Then clearly $T(PH) = T(\pi_2) \oplus V'$. Let $\kappa': T(PH) \rightarrow T(\pi_2)$ be the projection defined by this splitting. On the other hand PH is considered as a differentiable family of complex analytic principal

bundle over B . We may choose a differentiable family $\tilde{\theta}$ of type $(1, 0)$ connections on PH . So $\tilde{\theta}$ is just defined on each fibers and depends smoothly on the base space B . We define a connection ψ of the G -bundle $PH \rightarrow MH$ by $\psi = \tilde{\theta} \circ \kappa'$. Let K , $\tilde{\Theta}$, and Ψ be the curvature forms of κ , $\tilde{\theta}$, and ψ respectively. Let $\lambda: E \times P \rightarrow PH$ be the projection and $\lambda(e, p) = q$. Clearly $d\lambda(V \oplus 0) = V'$.

Lemma 4.3. For $X, Y \in T_q(\pi_2)$, $\Psi(X, Y) = \tilde{\Theta}(X, Y)$.

Lemma 4.4. For $X, Y \in V'_q$, $\Psi(X, Y) = \tilde{\theta}(d\lambda_{(e,p)}K(X_\#, Y_\#)_*)$ where, for $X \in h(M)$, $X_\#$ denotes the basic vector field of E .

Lemma 4.5. For $X \in T_q(\pi_2)$ of type $(0, 1)$ and $Y \in V'$, $\Psi(X, Y) = 0$.

Lemma 4.3 follows immediately from $\Psi = d\psi + \frac{1}{2}[\psi, \psi]$. For $X = d\lambda(X_\#)$ and $Y = d\lambda(Y_\#)$, we also have $\Psi(X, Y) = -\tilde{\theta}(\kappa'[X, Y])$. On the other hand since

$$\kappa([X_\#, Y_\#] - (\kappa[X_\#, Y_\#])_*) = 0 \quad \text{and} \quad K(X_\#, Y_\#) = -\kappa[X_\#, Y_\#]$$

we have $\kappa'(d\lambda([X_\#, Y_\#] + K(X_\#, Y_\#)_*)) = 0$. Hence

$$\begin{aligned} \Psi(X, Y) &= -\tilde{\theta}(\kappa'[X, Y]) = -\tilde{\theta}(\kappa'(d\lambda[X_\#, Y_\#])) \\ &= \tilde{\theta}(\kappa'd\lambda(K(X_\#, Y_\#)_*)) = \tilde{\theta}(d\lambda(K(X_\#, Y_\#)_*)). \end{aligned}$$

This proves Lemma 4.4.

We now assume that X is a section of $T(\pi_2)$ of type $(0, 1)$ and that $Y = d\lambda(Y_\#)$. Let ξ_t be the flow generated by $Y_\#$. We consider a trivialization $U \times H$ of E on an open set $U \subset B$. We may write $\xi_t(b, h) = (\xi_t^1(b), \xi_t^2(h))$ for $b \in U$ and $h \in H$. Putting $\xi_t^2(1) = \rho_t$, by the right invariance of ξ_t we have $\xi_t^2(h) = \rho_t h$. Let $U \times P$ be a trivialization of $PH \times B$. Then $\lambda: E \times P \rightarrow PH$ is given over U by $\lambda: U \times H \times P \rightarrow U \times P$, $\lambda(b, h, p) = (b, hp)$. Therefore the flow η_t generated by $Y = d\lambda(Y_\#)$ is expressed by $\eta_t(b, p) = (\xi_t^1(b), \rho_t p)$. Since ρ_t is an automorphism of P , $\eta_{t,*}(X)$ is also a section of $T(\pi_2)$ of type $(0, 1)$. Therefore

$$[X, Y]_q = \lim_{t \rightarrow 0} t^{-1}((\eta_{t,*}Y)_q - Y_q)$$

is also type $(0, 1)$. Then we obtain Lemma 4.5 from

$$\Psi(X, Y) = -\tilde{\theta}(\kappa'[X, Y]) = -\tilde{\theta}([X, Y]) = 0.$$

Returning to the proof of Theorem 4.1, the curvature form Ψ restricted to a fiber does not have type $(2, 0)$ part. This fact together with Lemma 4.5 shows that only the $(1, 1)$ part of $\tilde{\Theta}$ contributes to the integration over the fiber of $\phi(\Psi)$, $\phi \in I^{n+k}(G)$. Thus we obtain from Lemma 4.3, Lemma 4.4 and

Proposition 2.1 that

$$\begin{aligned} \pi_*\phi(\Psi) &= \binom{n+k}{k} \int_M \phi\left(\frac{i}{2\pi}\theta(K), \dots, \frac{i}{2\pi}\theta(K), \frac{i}{2\pi}\Theta, \dots, \frac{i}{2\pi}\Theta\right) \\ &= \left(\frac{i}{2\pi}\right)^k f_\phi(K). \end{aligned}$$

This proves Theorem 4.1.

Now let H^δ be the same group as H but equipped with the discrete topology. As before let $EH^\delta \rightarrow BH^\delta$ be the universal H^δ -bundle and put $MH^\delta = EH^\delta \times_{H^\delta} M$. The structure group of the bundle $M \rightarrow MH^\delta \rightarrow BH^\delta$ is the discrete group H^δ which acts on M holomorphically. Hence MH^δ admits a complex foliation \mathcal{F}_M of codimension n whose leaves are transverse to the fibers. The normal bundle $\nu(\mathcal{F}_M)$ of \mathcal{F}_M is naturally isomorphic to the subbundle of $T(MH^\delta)$ consisting of vectors which are tangent to the fibers. We can define a homomorphism $S: I_0^{n+k}(\text{GL}(n, \mathbb{C})) \rightarrow H^{2n+2k-1}(MH^\delta; \mathbb{C}/\mathbb{Z})$ as follows. For an element $\phi \in I_0^{n+k}(\text{GL}(n, \mathbb{C}))$, $S(\phi) \in H^{2n+2k-1}(MH^\delta; \mathbb{C}/\mathbb{Z})$ is the Simons class [10] defined by applying the Bott vanishing theorem to $\nu(\mathcal{F}_M)$. On the other hand consider the element $\Phi_0(\phi) \in I^k(H(M))$, where $\Phi_0: I_0^{n+k}(\text{GL}(n, \mathbb{C})) \rightarrow I^k(H(M))$ is the restriction of Φ . By Theorem 4.1 the cohomology class $W\Phi_0(\phi)$ is equal to $\pi_*W(\phi)$. Hence it is the reduction of the integral cohomology class $\pi'_*W(\phi) \in H^{2k}(BH; \mathbb{Z})$, where $\pi'_*: H^{2n+2k}(MH; \mathbb{Z}) \rightarrow H^{2k}(BH; \mathbb{Z})$ is the Gysin map. Now $EH^\delta \rightarrow BH^\delta$ is a flat H -bundle so that $W\Phi_0(\phi) = 0$ in $H^{2k}(BH^\delta; \mathbb{C})$. Hence we can define the Simons class $S_{\Phi_0(\phi), \pi_*W(\phi)} \in H^{2k-1}(BH^\delta; \mathbb{C}/\mathbb{Z})$. The above procedure defines a homomorphism $\mu: \text{Image } \phi_0 \rightarrow H^{2k-1}(BH^\delta; \mathbb{C}/\mathbb{Z})$ and we have

Corollary 4.6. *The following diagram commutes:*

$$\begin{array}{ccc} I_0^{n+k}(\text{GL}(n, \mathbb{C})) & \xrightarrow{\Phi_0} & \text{Image } \Phi_0 \\ \downarrow S & & \downarrow \mu \\ H^{2n+2k-1}(MH^\delta; \mathbb{C}/\mathbb{Z}) & \xrightarrow{\pi_*} & H^{2k-1}(BH^\delta; \mathbb{C}/\mathbb{Z}). \end{array}$$

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